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Heat transfer from a sphere in a stream of small Reynolds number

By P. L. RIMMER
Department of Mathematics, The University, Manchester 13, England

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The present paper deals with the temperature field past an isothermal sphere. Acrivos & Taylor (1962) have obtained results for the case $R \ll 1$ with no restriction on the Prandtl number. The results of the present paper are for the case $R < 1$ in which the Prandtl number is of order unity. An expansion for the Nusselt number is given up to and including the term of order R^2.

1. Introduction

The problem of heat transfer from a sphere in a flow of small Reynolds number is not a new one. A recent contribution to this subject is one reported by Acrivos & Taylor (1962) in which the velocity field is described by Stokes theory of creeping flow. These authors show, by means of matched asymptotic expansions, how the Nusselt number N can be obtained as an expansion in terms of the Péclet number σR, which is the product of the Prandtl number σ and the Reynolds number R. This expansion is valid for the case $R \ll 1$ and $\sigma R < 1$ and therefore is applicable to the flows of all fluids at extremely small Reynolds numbers.

Previously, by using a similar procedure, Proudman & Pearson (1957) had obtained an expansion for the velocity field for the flow past a sphere at small Reynolds numbers, an expansion in which as might be expected, Stokes solution is the leading term. The present paper is very similar to the work of Acrivos & Taylor but takes into account the extra terms in the velocity field predicted by Proudman & Pearson. It will be found that the Nusselt number now varies with R and σ separately and that the effect of the extra terms is to modify the term in $\sigma^2 R^2$ in the expansion for the Nusselt number obtained by Acrivos & Taylor. This expansion will be valid for the case $R < 1$ and σ of $O(1)$ and therefore will be applicable to the flows of all gases and certain liquids at Reynolds number less than unity. The form of the energy equation (2.1) used in the solution imposes further restrictions on the problem, namely that the fractional temperature difference $|T_w - T_\infty|/T_\infty$ must be large compared with the square of the Mach number and, by neglecting changes in density, we require also that the fractional temperature difference must be small compared with unity. This implies that the speed of the flow must be small compared with the speed of sound which is compatible with the restriction to small Reynolds numbers.

The method of solution is to obtain the temperature field as an expansion for $R < 1$ in each of two regions, one close to, and the other far from the sphere.

The inner (or Stokes) solution satisfies the boundary condition on the sphere, the outer (or Oseen) solution satisfies the boundary condition at infinity and the two solutions are made to match each other in the usual manner.

The results required from Proudman & Pearson are as follows. In the inner region the expansion for the non-dimensional stream function is

$$\psi(r,\theta) = \tfrac{1}{4}(2r^2-3r+r^{-1})(1-\mu^2)+R\{\tfrac{3}{32}(2r^2-3r+r^{-1})(1-\mu^2) - \tfrac{3}{32}(2r^2-3r+1-r^{-1}+r^{-2})\mu(1-\mu^2)\}+O(R^2\log R), \quad (1.1)$$

where r is the distance from the centre of the sphere divided by the radius a, θ is the angle from the direction of the undisturbed stream, $\mu = \cos\theta$, and $R = Ua/\nu$, in which U is the velocity of the incident stream and ν is the kinematic viscosity. In the outer region new variables are defined by

$$\rho = \sigma R r, \quad \Psi = \sigma^2 R^2 \psi,$$

and the stream function is given by the expansion

$$\Psi(\rho,\theta) = \tfrac{1}{2}\rho^2(1-\mu^2)-\sigma^2 R\{\tfrac{3}{2}(1+\mu)[1-\exp\{-\tfrac{1}{2}\rho\sigma^{-1}(1-\mu)\}]\}+O(R^2). \quad (1.2)$$

The factor σ does not appear in Proudman & Pearson's analysis, but it has been introduced here because ρ, rather than $\sigma^{-1}\rho$ is the variable required in the outer version of the energy equation.

2. Basic equations

The basic equation for the transport of energy is, in dimensionless form,

$$\nabla_r^2 t = \sigma R\mathbf{u}\,.\,\mathrm{grad}\, t, \quad (2.1)$$

in terms of inner variables, where t is the normalized temperature such that the boundary conditions are

$$t = 1 \quad \text{at} \quad r = 1, \quad t \to 0 \quad \text{as} \quad r \to \infty$$

and where ∇_r^2 is the operator

$$\nabla_r^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial}{\partial \mu}\right\}.$$

In the inner region we solve (2.1) with the boundary condition $t = 1$ at $r = 1$. In the outer region the convective and diffusive terms in (2.1) are of the same order of magnitude and so we introduce new variables in this region given by

$$\rho = \sigma R r, \quad T(\rho,\mu) = t(r,\mu),$$

and (2.1) becomes

$$\nabla_\rho^2 T = \mathbf{u}\,.\,\mathrm{grad}\, T. \quad (2.2)$$

In the outer region we solve equation (2.2) for T with the boundary condition $T \to 0$ as $\rho \to \infty$ and require that this solution matches with the solution $t(r,\mu)$ valid in the inner region.

We assume next that the inner and outer expansions may be represented, respectively, by

$$t(r,\mu) = \sum_{n=0}^{\infty} f_n(R)\, t_n(r,\mu), \quad f_0(R) = 1$$

and

$$T(\rho,\mu) = \sum_{n=0}^{\infty} F_n(R)\, T_n(\rho,\mu),$$

where

$$\lim_{R\to 0} \frac{f_{n+1}}{f_n} = 0, \quad \lim_{R\to 0} \frac{F_{n+1}}{F_n} = 0.$$

3. Construction of the solution

(i) *First expansion term*

As the first approximation we put $R = 0$ which gives as the differential equation for $t_0(r,\mu)$

$$\nabla_r^2 t_0 = 0.$$

The equation for $T_0(\rho,\mu)$ is

$$\nabla_\rho^2 T_0 = \partial T_0/\partial z$$

where $z = \rho\mu$. Acrivos & Taylor give the required solutions, namely

$$t_0 = r^{-1}, \tag{3.1}$$

$$T_0(\rho,\mu) = \rho^{-1}\exp\{\tfrac{1}{2}\rho(\mu-1)\}, \tag{3.2}$$

and $F_0(R) = \sigma R$.

(ii) *Second expansion term*

For small ρ

$$T_0(\rho,\mu) \sim \rho^{-1} - \tfrac{1}{2}(1-\mu) + \ldots,$$

and to satisfy the matching condition we expect, for large r,

$$t(r,\mu) \sim r^{-1} - \tfrac{1}{2}\sigma R(1-\mu) + \ldots.$$

In view of this we put $f_1(R) = \sigma R$. The velocity in the inner region is

$$\mathbf{u} = \mathbf{u}_0(r,\mu) + R\mathbf{u}_1(r,\mu) + O(R^2 \log R), \tag{3.3}$$

where $\mathbf{u}_0$ and $\mathbf{u}_1$ are obtained from (1.1).
Therefore the equation for $t_1(r,\mu)$ is

$$\nabla_r^2 t_1 = \mathbf{u}_0 \cdot \operatorname{grad} t_0,$$

which becomes

$$\nabla_r^2 t_1 = (-r^{-2} + \tfrac{3}{2}r^{-3} - \tfrac{1}{2}r^{-5})\mu.$$

As in Acrivos & Taylor's paper the required solution is

$$t_1(r,\mu) = -\tfrac{1}{2}(1-r^{-1}) + \tfrac{1}{8}(4 - 6r^{-1} + 3r^{-2} - r^{-3})\mu.$$

From the forms of t_0 and t_1 and in order that the matching condition be satisfied we expect that, for small ρ,

$$T(\rho,\mu) \sim \sigma R(\rho^{-1} - \tfrac{1}{2} + \tfrac{1}{2}\mu) + \sigma^2 R^2(\tfrac{1}{2} - \tfrac{3}{4}\mu)\rho^{-1} + \ldots.$$

In view of this we put $F_1(R) = \sigma^2 R^2$. The velocity in the outer region is

$$\mathbf{u} = \mathbf{U}_0(\rho,\mu) + R\mathbf{U}_1(\rho,\mu) + O(R^2),$$

where $\mathbf{U}_0$ and $\mathbf{U}_1$ are obtained from (1.2). It is at this point that the present work diverges from that of Acrivos & Taylor due to the different forms of $\mathbf{U}_1(\rho,\mu)$ in the two cases.

The equation for $T_1(\rho,\mu)$ is

$$\nabla_\rho^2 T_1 - \frac{\partial T_1}{\partial z} = \sigma \mathbf{U}_1 \cdot \operatorname{grad} T_0. \tag{3.4}$$

If we put $T_1(\rho,\mu) = e^{\frac{1}{2}\rho\mu}T_1^*(\rho,\mu)$, equation (3.4) becomes

$$(\nabla_\rho^2 - \tfrac{1}{4})T_1^* = -\tfrac{3}{4}\rho^{-4}e^{-\frac{1}{2}\rho}[2\sigma + \sigma\rho - \sigma\rho\mu - \exp\{-\tfrac{1}{2}\rho\sigma^{-1}\}\{(\sigma\rho + 2\sigma + \rho)\exp\{\tfrac{1}{2}\rho\mu\sigma^{-1}\} - (\sigma - 1)\rho\mu\exp\{\tfrac{1}{2}\rho\mu\sigma^{-1}\}\}].$$

If we use
$$e^{\mu z} = (2\pi)^{\frac{1}{2}}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})z^{-\frac{1}{2}}I_{n+\frac{1}{2}}(z)P_n(\mu),$$

$$\mu e^{\mu z} = (2\pi)^{\frac{1}{2}}\sum_{n=0}^{\infty}n(n+\tfrac{1}{2})z^{-\frac{3}{2}}I_{n+\frac{1}{2}}(z)P_n(\mu) + (2\pi)^{\frac{1}{2}}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})z^{-\frac{1}{2}}I_{n+\frac{3}{2}}(z)P_n(\mu),$$

where $I_{n+\frac{1}{2}}(z)$ is a modified Bessel function and $P_n(\mu)$ is the Legendre polynomial and, if we put
$$T_1^*(\rho,\mu) = \sum_{n=0}^{\infty}g_n(\rho)P_n(\mu),$$
the equation satisfied by $g_0(\rho)$ is

$$\rho^{-2}\frac{d}{d\rho}\left(\rho^2\frac{dg_0}{d\rho}\right) - \tfrac{1}{4}g_0 = -\tfrac{3}{4}\rho^{-4}e^{-\frac{1}{2}\rho}[\sigma(2+\rho) + (\tfrac{1}{2}\pi)^{\frac{1}{2}}\exp\{-\tfrac{1}{2}\rho\sigma^{-1}\}\{-(\sigma\rho + 2\sigma + \rho)(2\sigma\rho^{-1})^{\frac{1}{2}}I_{\frac{1}{2}}(\tfrac{1}{2}\rho\sigma^{-1}) + (\sigma-1)\rho(2\sigma\rho^{-1})^{\frac{1}{2}}I_{\frac{3}{2}}(\tfrac{1}{2}\rho\sigma^{-1})\}]. \quad (3.5)$$

The complementary functions of equation (3.5) are $g_0 = \rho^{-1}e^{\pm\frac{1}{2}\rho}$, and a particular integral of this equation can be found by the method of variation of parameters. Since
$$I_{\frac{1}{2}}(z) = \tfrac{1}{2}(\tfrac{1}{2}\pi)^{-\frac{1}{2}}z^{-\frac{1}{2}}(e^z - e^{-z})$$
and
$$I_{\frac{3}{2}}(z) = \tfrac{1}{2}(\tfrac{1}{2}\pi)^{-\frac{1}{2}}z^{-\frac{1}{2}}\{(1 - z^{-1})e^z + (1 + z^{-1})e^{-z}\},$$
the solution of (3.5) which vanishes at infinity can be written, after straightforward but lengthy manipulation, as

$$g_0(\rho) = A\rho^{-1}e^{-\frac{1}{2}\rho} - \tfrac{1}{2}\sigma[-\tfrac{1}{2}\sigma^2\rho^{-3}(1-\rho)e^{-\frac{1}{2}\rho} + \tfrac{1}{2}\sigma\rho^{-3}\{\sigma - (\sigma-1)\rho\}e^{-\frac{1}{2}\rho - \rho\sigma^{-1}} + \sigma^{-1}\rho^{-1}e^{-\frac{1}{2}\rho}E_1(\rho\sigma^{-1}) - \tfrac{1}{2}\rho^{-1}(\sigma^2 - 3)e^{\frac{1}{2}\rho}E_1(\rho) + \tfrac{1}{2}\sigma^{-1}(\sigma+1)^2(\sigma-2)\rho^{-1}e^{\frac{1}{2}\rho}E_1\{\sigma^{-1}(\sigma+1)\rho\}],$$
where
$$E_1(x) = \int_x^{\infty}t^{-1}e^{-t}\,dt$$
and A is a constant of integration.

For small x,
$$E_1(x) = -\gamma - \log x + x + O(x^2).$$
Therefore, for small ρ,

$$g_0(\rho) = \rho^{-1}\{A + \tfrac{1}{8}\sigma(1 - 2\sigma) + \tfrac{1}{4}(\sigma+1)^2(\sigma-2)\log(\sigma+1) - \tfrac{1}{4}\sigma(\sigma^2 - 3)\log\sigma\} + \{-\tfrac{1}{2}A - \tfrac{1}{2}(\gamma + \log\rho) - \tfrac{1}{8}(\sigma^3 - 3\sigma - 4)\log\sigma + \tfrac{1}{8}(\sigma+1)^2(\sigma-2)\log(\sigma+1) - \tfrac{1}{48}(6\sigma^2 - 3\sigma - 56)\} + O(\rho).$$

Matching this part of the solution with $t(r,\mu)$ we find that
$$A = \tfrac{1}{8}(2\sigma^2 - \sigma + 4) - \tfrac{1}{4}(\sigma+1)^2(\sigma-2)\log(\sigma+1) + \tfrac{1}{4}\sigma(\sigma^2 - 3)\log\sigma.$$

Although we have obtained only $g_0(\rho)$ explicitly we shall find that this gives enough information about $T_1(\rho,\mu)$ to obtain the next two terms of the inner expansion completely.

Up to this point we have, for small ρ,

$$T(\rho,\mu) \sim \sigma R(\rho^{-1}-\tfrac{1}{2})+\sigma^2R^2[\tfrac{1}{2}\rho^{-1}-\tfrac{1}{2}\log\rho$$
$$+\{-\tfrac{1}{24}(6\sigma^2-3\sigma-22)+\tfrac{1}{4}(\sigma+1)^2(\sigma-2)\log(\sigma+1)$$
$$-\tfrac{1}{4}(\sigma^3-3\sigma-2)\log\sigma-\tfrac{1}{2}\gamma\}+O(\rho)]+\text{terms involving } P_n(\mu) \quad \text{for} \quad n\geqslant 1. \quad (3.6)$$

To satisfy the matching condition we expect, for large r,

$$t(r,\mu) \sim r^{-1}+\sigma R(\tfrac{1}{2}r^{-1}-\tfrac{1}{2})+\sigma^2R^2\log\sigma R(-\tfrac{1}{2})$$
$$+\sigma^2R^2(-\tfrac{1}{2}\log r+\text{constant})+O(\sigma^3R^3)+\text{terms involving } P_n(\mu) \quad \text{for} \quad n\geqslant 1. \quad (3.7)$$

(iii) *Higher-order expansion terms*

From the form of equation (3.7) we expect the next two terms in the inner expansion to be of order $\sigma^2R^2\log\sigma R$ and σ^2R^2 respectively. This is indeed the case but, as usual in such cases, these two terms are not independent. We must take the sum of these two terms as the next approximation in the inner expansion.

On putting $f_2(R) = \sigma^2R^2\log\sigma R$, $f_3(R)=\sigma^2R^2$, the equation for $t_2(r,\mu)$ is found to be

$$\nabla_r^2 t_2 = 0.$$

The solution of this equation which vanishes at $r=1$, is of the correct order in the outer region, and agrees with equation (3.7) is

$$t_2(r,\mu) = -\tfrac{1}{2}(1-r^{-1}).$$

The equation for $t_3(r,\mu)$ is

$$\nabla_r^2 t_3 = \mathbf{u}_0\,.\,\mathrm{grad}\,t_1+\sigma^{-1}\mathbf{u}_1\,.\,\mathrm{grad}\,t_0.$$

This becomes

$$\nabla_r^2 t_3 = \tfrac{1}{48}(16r^{-1}-24r^{-2}+7r^{-4}+6r^{-5}-9r^{-6}+4r^{-7})\,P_0(\mu)$$
$$-\tfrac{1}{16}(4+3\sigma^{-1})(2r^{-2}-3r^{-3}+r^{-5})\,P_1(\mu)-\tfrac{1}{48}(16r^{-1}-60r^{-2}+90r^{-3}-65r^{-4}$$
$$+15r^{-5}+9r^{-6}-5r^{-7})\,P_2(\mu)+\tfrac{3}{16}\sigma^{-1}(2r^{-2}-3r^{-3}+r^{-4}-r^{-5}+r^{-6})\,P_2(\mu).$$

The general solution of this equation is

$$t_3(r,\mu) = \tfrac{1}{48}(8r+\tfrac{7}{2}r^{-2}+r^{-3}-\tfrac{3}{4}r^{-4}+\tfrac{1}{5}r^{-5}-24\log r)\,P_0(\mu)$$
$$-\tfrac{1}{16}(4+3\sigma^{-1})(-1+\tfrac{3}{2}r^{-1}+\tfrac{1}{4}r^{-3})\,P_1(\mu)-\tfrac{1}{48}(-4r+10-15r^{-1}+\tfrac{65}{4}r^{-2}$$
$$+\tfrac{3}{2}r^{-4}-\tfrac{5}{14}r^{-5}-3r^{-3}\log r)\,P_2(\mu)+\tfrac{3}{16}\sigma^{-1}(-\tfrac{1}{3}+\tfrac{1}{2}r^{-1}-\tfrac{1}{4}r^{-2}+\tfrac{1}{6}r^{-4}$$
$$+\tfrac{1}{5}r^{-3}\log r)\,P_2(\mu)+\sum_{n=0}^{\infty}(A_nr^n+B_nr^{-n-1})\,P_n(\mu),$$

where the A_n and B_n are constants.

Matching this solution with the known part of the outer solution and using the condition $t_3 = 0$ at $r=1$ we have

$$A_0 = -\tfrac{1}{24}(6\sigma^2-3\sigma-22)+\tfrac{1}{4}(\sigma+1)^2(\sigma-2)\log(\sigma+1)-\tfrac{1}{4}(\sigma^3-3\sigma-2)\log\sigma-\tfrac{1}{2}\gamma,$$
$$A_0+B_0 = -\tfrac{239}{960},$$
$$A_1 = -\tfrac{1}{4}, \quad B_1 = \tfrac{1}{64}(28+9\sigma^{-1}),$$
$$B_2 = \tfrac{1}{48}(\tfrac{235}{28}-\tfrac{3}{4}\sigma^{-1}),$$

$A_n = 0$ for $n\geqslant 2$, $B_n = 0$ for $n \geqslant 3$. Therefore $t_3(r,\mu)$ has been determined completely.

From the form of the basic equations and the actual solutions obtained so far it would appear that successive terms in the expansions are given by

$$F_2(R) = \sigma^3R^3\log\sigma R,\ F_3(R) = \sigma^3R^3,\ f_4(R) = \sigma^3R^3\log\sigma R,\ f_5(R) = \sigma^3R^3, \text{ etc.}$$

Justification for these expressions can only be given after the corresponding solutions have been obtained and the algebraic effort involved in their calculation increases rapidly at each successive stage.

4. Conclusion

Acrivos & Taylor used Stokes's formula for the velocity field, which gives a good approximation to the flow past a sphere for small Reynolds numbers. Because they ignored terms of order R in the velocity, as is appropriate when $R \ll 1$, their results differ from those of the present work in the term of order σ^2R^2 in the temperature field.

The main result of interest here is an expression for the average Nusselt number N which, in non-dimensional variables, is given by

$$N = -\int_{-1}^{1}\left(\frac{\partial t}{\partial r}\right)_{r=1} d\mu.$$

From the inner expansion solution we have

$$N = N_1 = 2+\sigma R+\sigma^2R^2\log\sigma R+f(\sigma)\,\sigma^2R^2+\ldots,$$

where

$$f(\sigma) = \tfrac{1}{4}\{(2\sigma^2-\sigma+4\gamma-\tfrac{213}{40})+2(\sigma^3-3\sigma-2)\log\sigma-2(\sigma+1)^2(\sigma-2)\log(\sigma+1)\}.$$

This can be compared with the results of Acrivos & Taylor which can be written in the form

$$N = N_2 = 2+\sigma R+\sigma^2R^2\log\sigma R+0{\cdot}8293\sigma^2R^2+\tfrac{1}{2}\sigma^3R^3\log\sigma R+\ldots.$$

σ	$f(\sigma)$	σ	$f(\sigma)$
0·50	1·0999	1·05	0·8703
0·55	1·0644	1·10	0·8592
0·60	1·0336	1·15	0·8490
0·65	1·0066	1·20	0·8395
0·70	0·9828	1·25	0·8306
0·75	0·9615	1·30	0·8223
0·80	0·9424	1·35	0·8146
0·85	0·9252	1·40	0·8074
0·90	0·9096	1·45	0·8006
0·95	0·8953	1·50	0·7941
1·00	0·8823		

TABLE 1

Table 1 plots the value of $f(\sigma)$ for $0{\cdot}5 \leqslant \sigma \leqslant 1{\cdot}5$ and it can be seen that the coefficients of σ^2R^2 in the two Nusselt number expansions agree for a value of σ of approximately 1·25.

Table 2 plots the two expressions N_1 and N_2 in the range $0 < R \leqslant 1$ for $\sigma = 0{\cdot}70$ and in this range the two expressions differ by less than five per cent.

$\sigma = 0{\cdot}70$

R	N_1	N_2
0·05	2·0318	2·0321
0·10	2·0606	2·0618
0·15	2·0880	2·0910
0·20	2·1150	2·1207
0·25	2·1423	2·1517
0·30	2·1705	2·1845
0·35	2·2000	2·2196
0·40	2·2312	2·2572
0·45	2·2646	2·2979
0·50	2·3005	2·3418
0·55	2·3392	2·3892
0·60	2·3811	2·4403
0·65	2·4266	2·4954
0·70	2·4759	2·5547
0·75	2·5294	2·6183
0·80	2·5873	2·6864
0·85	2·6501	2·7591
0·90	2·7180	2·8367
0·95	2·7913	2·9192
1·00	2·8704	3·0068

TABLE 2

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